On Taylor series expansion of $(1+z)^A$ for $|z| > 1$

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Abstract

It is well known that the Taylor series expansion of $(1+z)^A$ does not converge for $|z| > 1$ where A is a real number which is not equal to zero or a positive integer. A limited series expansion of this expression is obtained in this paper for $|z| > 1$ as a product of convergent series.

Keywords:

1. Introduction

It is well known that the Taylor series expansion of $(1+z)^A$ is given by

$$(1+z)^A = \sum_{n=0}^{\infty} \binom{A}{n} z^n \tag{1}$$

for $|z| < 1$ where $\binom{A}{n}$ is the binomial choose function. It is well known that the series expansion does not converge for $|z| > 1$ where A is a real number which is not equal to zero or a positive integer.

We could obtain a limited series expansion for $|z| > 1$ by writing the above expression as follows

$$(1+z)^A = (1+\frac{z}{2}+\frac{z}{2})^A = (1+\frac{z}{2})^A(1+\frac{\frac{z}{2}}{(1+\frac{z}{2})})^A = (1+\frac{z}{2})^A(1+\frac{z}{z+2})^A \tag{2}$$

The second term in the above equation has a convergent series representation, given that $|\frac{z}{z+2}| < 1$. If $|\frac{z}{2}| > 1$, we can write

$$(1+\frac{z}{2})^A = (1+\frac{z}{4})^A(1+\frac{z}{z+4})^A \tag{3}$$

Repeating this procedure iteratively, if m_0 is the minimum value for which $|\frac{z}{2^{m_0}}| < 1$, we can write

$$(1+z)^A = (1+\frac{z}{2^{m_0}})^A \prod_{r=1}^{m_0}(1+\frac{z}{z+2^r})^A \tag{4}$$

Each of the terms in the above product of terms has a convergent series representation. Given that we can write the convergent series expansion for each of the terms above as $(1+\frac{z}{2^{m_0}})^A =$

$\sum_{n=0}^{\infty}\binom{A}{n}(\frac{z}{2^{m_0}})^n$ and $(1+\frac{z}{(z+2^r)})^A = \sum_{m=0}^{\infty}\binom{A}{m}(\frac{z}{(z+2^r)})^m$, where $\binom{A}{n}$ represents the Choose function[1], we have the **series expansion for** $(1+z)^A$ **expressed as a product of convergent series, which converges for** $|z| > 1$ as follows:

$$(1+z)^A = [\sum_{n=0}^{\infty}\binom{A}{n}(\frac{z}{2^{m_0}})^n][\prod_{r=1}^{m_0}\sum_{m=0}^{\infty}\binom{A}{m}[\frac{z}{z+2^r}]^m] \tag{5}$$

2. Section 2

Let us take the case of $m_0 = 1$ for $1 < |z| < 2$. Expanding the term $\frac{1}{(z+2^r)^m}$ in the above equation 5 as Taylor series around a point $z = 0$, we have for $m > 0$

$$\frac{1}{(z+2^r)^m} = \sum_{j=0}^{\infty} b(j,r,m)z^j \tag{6}$$

where $b(0,r,m) = \frac{1}{(2^r)^m}$ and $b(j,r,m)$ is given as follows for $j = 1,2,3...$

$$b(j,r,m) = \binom{m+j-1}{j}\frac{(-1)^j}{(2^r)^{m+j}}; \tag{7}$$

For $m = 0$, $\frac{1}{(z+2^r)^m} = 1$. Now we can write the the series expansion of $(1+z)^A$ which converges for $1 < |z| < 2$, as a product of terms expanded in Taylor series as follows:

$$(1+z)^A = [\sum_{n=0}^{\infty}\binom{A}{n}(\frac{z}{2})^n][\sum_{m=0}^{\infty}\binom{A}{m}z^m\sum_{j=0}^{\infty}b(j,1,m)z^j] \tag{8}$$

For the case of $m_0 > 1$ for $|z| > 2$, we can write as follows:

$$(1+z)^A = [\sum_{n=0}^{\infty}\binom{A}{n}(\frac{z}{2^{m_0}})^n][\sum_{m=0}^{\infty}\binom{A}{m}z^m\sum_{j=0}^{\infty}b(j,m_0,m)z^j][\prod_{r=1}^{m_0-1}\sum_{m=0}^{\infty}\binom{A}{m}z^m[\frac{1}{z+2^r}]^m] \tag{9}$$

The last term in the above equation $(\frac{1}{z+2^r})^m$ can be expressed as follows:

$$(\frac{1}{z+2^r})^m = (z+2^r)^{-m} = 2^{-r*m}(1+\frac{z}{2^r})^{-m} \tag{10}$$

The term$(1+\frac{z}{2^r})^{-m}$ can be recursively expanded using Eq.9 by substituting $z \to \frac{z}{2^r}$ and $A \to -m$ and $m_0 \to m_0 - r$ to obtain the series expansion of $(1+z)^A$ which converges for $|z| > 1$ as a product of terms expanded in Taylor series.

3. Section 3

Let us consider the following binomial expression

$$(x+y)^A \tag{11}$$

where A is a real number which is not equal to zero or a positive integer and $z = \frac{x}{y}$ and $|z| > 1$. Writing $(x+y)^A = (1+\frac{x}{y})^A y^A = (1+z)^A y^A$, we can write the series expansion of this expression using results obtained in equations 5 and 9 as follows:

$$(x+y)^A = y^A \sum_{n=0}^{\infty} \binom{A}{n} (\frac{z}{2^{m_0}})^n \prod_{r=1}^{m_0} \sum_{m=0}^{\infty} \binom{A}{m} [\frac{z}{z+2^r}]^m \tag{12}$$

$$(x+y)^A = y^A [\sum_{n=0}^{\infty} \binom{A}{n} (\frac{z}{2^{m_0}})^n][\sum_{m=0}^{\infty} \binom{A}{m} z^m \sum_{j=0}^{\infty} b(j, m_0, m) z^j][\prod_{r=1}^{m_0-1} \sum_{m=0}^{\infty} \binom{A}{m} z^m [\frac{1}{z+2^r}]^m] \tag{13}$$

4. Conclusion

It has been shown that the Taylor series expansion of $(1+z)^A$ can be expanded as a product of convergent series, for $|z| > 1$ where A is a real number which is not equal to zero or a positive integer.

5. Acknowledgements

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6. References

[1] Abramowitz, M. and Stegun, I. A. (Eds.). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, 1972.